Vertical Asymptotes and Infinite Limits (Section 1.7)

Let $f(x) = \dfrac{1}{(x-2)}$

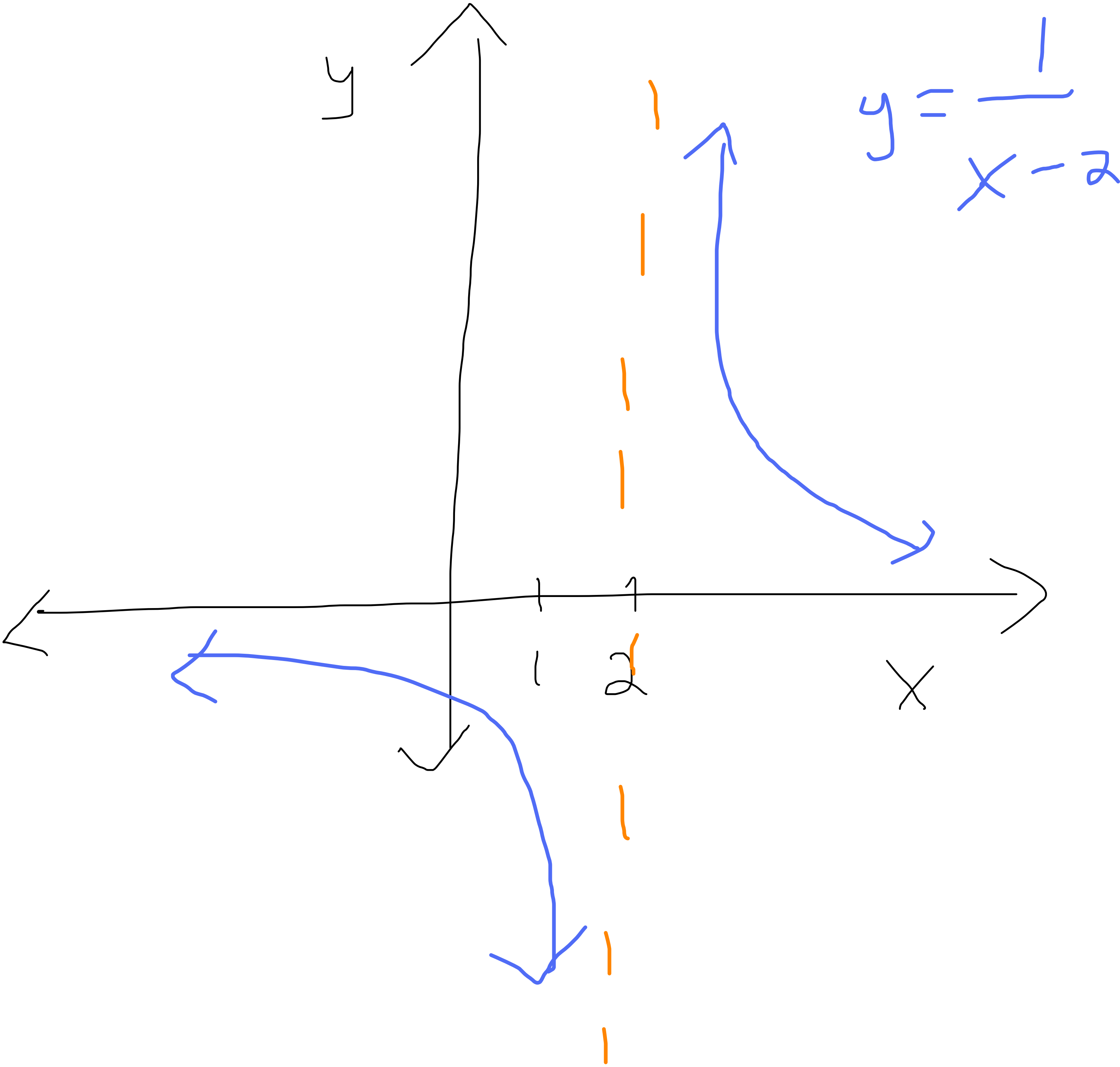
Plugging in $x=2$ gives $\frac{1}{0}$,

so $\lim_{x \to 2^+} \dfrac{1}{x-2}$ does not

exist, but it does not

exist in a special way.

Picture



As we get closer to $x=2$ from the right, the values of $\frac{1}{x-2}$ become larger than any number we choose

We then say $\frac{1}{x-2}$ has a vertical asymptote at $x=2$

and write $\lim_{x \to 2^+} \frac{1}{x-2} = \infty.$

Similarly, as we get closer to $x=2$ from the left, the values of $\frac{1}{x-2}$ become less than the negative of any number we choose. We then write

$$\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$$

The same definitions work for an arbitrary function f at a point $x=a$. If $\boxed{f(a)=\frac{k}{0}}$

where $k \neq 0$, then f has a vertical asymptote at $x=a$. If

$$\lim_{x\to a^+} f(x) = \infty = \lim_{x\to a^-} f(x),$$

we write $\lim\limits_{x\to a} f(x) = \infty$

If $\lim_{x \to a^-} f(x) = -\infty = \lim_{x \to a^+} f(x)$,

we write $\lim_{x \to a} f(x) = -\infty$.

If the limits are unequal,

we say $\lim_{x \to a} f(x)$ does not exist.

Note: $\lim_{x \to 2} \frac{1}{x-2}$ does not exist.

Example 1: Find all vertical asymptotes for

$$f(x) = \frac{1}{(x-4)^2(x-3)}.$$

At each one, determine whether the limit is ∞, $-\infty$, or does not exist.

We see that $f(4) = f(3) = \frac{1}{0}$, so that f has vertical asymptotes at $\boxed{x=4}$ and $\boxed{x=3}$. Check $x=4$ first

$$\lim_{x \to 4^+} \frac{1}{(x-4)^2(x-3)}$$

Since $(x-4)^2 \geq 0$ always and $x-3$ is positive close to $x=4$,

$$\boxed{\lim_{x \to 4^+} \frac{1}{(x-4)^2(x-3)} = \infty}$$

Using the same reasoning we get $\boxed{\lim_{x \to 4^-} \frac{1}{(x-4)^2(x-3)} = \infty,}$

so $\boxed{\lim_{x \to 4} \frac{1}{(x-4)^2(x-3)} = \infty}$

Now check $x=3$

$$\lim_{x\to 3^+} \frac{1}{(x-4)^2(x-3)}$$

Again $(x-4)^2 \geq 0$, and

$\lim_{x\to 3^+}$ means $x>3$, so $x-3>0$

This shows $\boxed{\lim_{x\to 3^+} \frac{1}{(x-4)^2(x-3)} = \infty}$

For $\lim_{x\to 3^-} \frac{1}{(x-4)^2(x-3)}$, $(x-4)^2 \geq 0$

but now, $x<3$, so $x-3<0$.

This means $\boxed{\lim_{x\to 3^-} \frac{1}{(x-4)^2(x-3)} = -\infty}$

Since these limits are not equal,

$$\lim_{x \to 3} \frac{1}{(x-4)^2(x-3)} \text{ does not exist.}$$

To summarize, the vertical asymptotes are $x=3$ and $x=4$,

$$\lim_{x \to 4} \frac{1}{(x-4)^2(x-3)} = \infty, \text{ but}$$

$$\lim_{x \to 3} \frac{1}{(x-4)^2(x-3)} \text{ does not exist}$$

Example 2: Find all vertical asymptotes for

$$f(x) = \frac{7x^2 - 21x + 14}{(x-3)(x-1)^2}$$

Note the zeros of the denominator are $x = 1$ and $x = 3$. Plugging these into f,

$$f(3) = \frac{63 - 63 + 14}{0} = \frac{14}{0}$$

so f has a vertical asymptote at $\boxed{x = 3}$

However,

$$f(1) = \frac{7-21+14}{0} = \frac{0}{0}$$

which means more work!

Really the only thing we can do is factor the numerator.

$$f(x) = \frac{7x^2-21x+14}{(x-1)^2(x-3)}$$

$$= \frac{7(x^2-3x+2)}{(x-1)^2(x-3)}$$

$$= \frac{7(x-2)\cancel{(x-1)}}{(x-1)^{\cancel{2}}(x-3)}$$

$$= \frac{7(x-2)}{(x-1)(x-3)} \quad (\text{if } x \neq 1)$$

Plugging in $x=1$ now gives us

$$f(1) = \frac{-7}{0}, \text{ so}$$

f has a vertical asymptote at $\boxed{x=1}$.

So the vertical asymptotes are $\boxed{x=1}$ and $\boxed{x=3}$

Horizontal Asymptotes and Limits to Infinity:

(Section 3.4)

A function f has a horizontal asymptote at $y = L$ if either the values of f "tend to" L as x gets bigger and bigger (write $\lim\limits_{x \to \infty} f(x) = L$) or the values of f "tend to" L as $-x$ gets bigger and bigger (write $\lim\limits_{x \to -\infty} f(x) = L$)

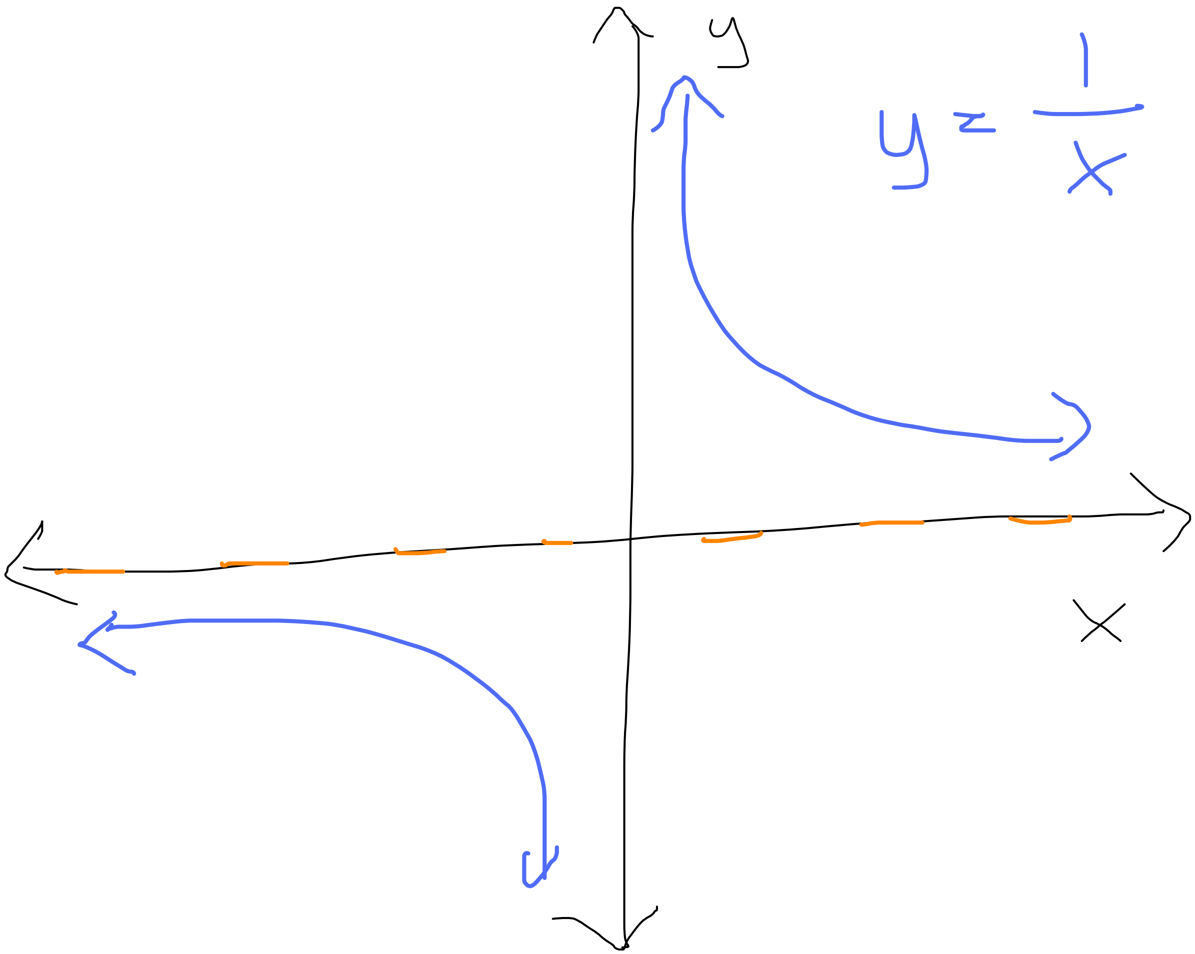
Very Important Example 3:

$f(x) = \frac{1}{x}$. Then

$$\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0$$

which means f has a horizontal asymptote at $\boxed{y=0}$

Picture



All other results follow from this one!

Example 4: Find

$$\lim_{x \to \infty} \frac{3x-7}{-13x+11}$$

Divide numerator and denominator by the highest power of x you see, which is $x = x^1$.

$$\frac{3x-7}{-13x+11} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \frac{\frac{3x-7}{x}}{\frac{-13x+11}{x}}$$

$$= \frac{3-\frac{7}{x}}{-13+\frac{11}{x}}$$

Now take the limit using the limit laws

$$\lim_{x\to\infty} \frac{3x-7}{-13x+11} = \lim_{x\to\infty} \frac{3-\frac{7}{x}}{-13+\frac{11}{x}}$$

$$= \frac{3-7\left(\lim_{x\to\infty} \frac{1}{x}\right)}{-13+11\left(\lim_{x\to\infty} \frac{1}{x}\right)} = \frac{3-(7\cdot 0)}{-13+(11\cdot 0)}$$

$$= \boxed{-\frac{3}{13}}$$